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POISSON LIMITS FOR A HARD-CORE CLUSTERING MODEL

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Let $X_{1n}, \ldots, X_{nn}$ denote the locations of n points in a bounded, γ-dimensional, Euclidean region D_n which has positive γ-dimensional Lebesgue measure $\mu(D_n)$. Let $\{Y_n(r): r>0\}$ be the interpoint distance process for these points where $Y_n(r)$ is the number of pairs of points (X_{in}, X_{jn}) which with $i<j$ have Euclidean distance $\|X_{in}-X_{jn}\|<r$. In this article we study the limiting distribution of $Y_n(r)$ when $n\to\infty$ and $\mu(D_n)\to\infty$, and the joint density of $X_{1n}, \ldots, X_{nn}$ is of the form

$$f(x_1,\ldots,x_n)=\begin{cases} C_n\exp(vy_n(r)) & \text{if } y_n(r_0)=0,\\ 0 & \text{if } y_n(r_0)>0\end{cases}$$

where r_0 is a positive constant and C_n is a normalizing constant. These joint densities modify the Strauss [11] clustering model densities by introducing a hard-core component (no two points can have $\|X_{in}-X_{jn}\|<r_0$) found in the Matérn [4] models. In our main result we show that the interpoint distance process converges to a non-homogeneous Poisson process for r values in a bounded interval $0<r_0<r<r_{00}$ provided sparseness conditions discussed by Saunders and Funk [9] hold. The sparseness conditions which require $\mu(D_n)/n^2$ converges to a positive constant and the boundary of D_n is negligible are essentially equivalent to requiring that although the number of points n is large the region is large enough so that the points are sparse in this region. That is, it is rare for a point to have another point close to it. These results extend results for $v\leq 0$ given by Saunders and Funk [9] where it is shown that without the hard core component such results do not hold for $v>0$. Statistical applications are discussed.

Clustering model	radius of influence
hard-core	sparseness
Poisson process	weak convergence

1. Introduction

Let D_n be a γ-dimensional, bounded, Euclidean region with Lebesgue measure $\mu(D_n)>0$. The clustering model introduced by Strauss [11] and subsequently studied by Kelly and Ripley [4] assumes that the joint probability density function

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for n points $X_{1n}\in D_n,\ldots,X_{nn}\in D_n$ has the form

$$f_n(x_1,\ldots,x_n)=\begin{cases} e^{vy_n(r_1)}/M_n(v) & \text{if } x_1\in D_n,\ldots,x_n\in D_n,\\ 0 & \text{otherwise.}\end{cases}$$

Here, using $\|\cdot\|$ to denote Euclidean distance,

$$y_n(r_1)=\sum_{1\le i\le j\le n}\alpha_{r_1}(x_i,x_j),$$

$$\alpha_r(x_i,x_j)=\begin{cases}1 & \text{if } \|x_i-x_j\|<r,\\ 0 & \text{otherwise}\end{cases}$$

and $M_n(v)=E_0\{\exp(vY_n(r_1))\}$ is the moment generating function of $Y_n(r_1)$ evaluated at v and computed under the (i.i.u.d.) assumption that $X_{1n},\ldots,X_{nn}$ are independent and identically distributed uniformly on D_n.

The parameter r_1 is called the radius of influence and $Y_n(r_1)$, which represents the number of pairs of points within a distance r_1 of each other, is a measure of the clustering or repulsion of the points. The parameter v can be interpreted as follows. If $v=0$, then the points are i.i.u.d. If $v>0$, then clusters have higher probability than when $v=0$ while if $v<0$, then clusters have lower probability than when $v=0$. Hence $Y_n(r_1)$ typically has larger values when $v>0$ as opposed to $v=0$, and has smaller values when $v<0$ as opposed to $v=0$. In fact, the distribution of $Y_n(r_1)$ is stochastically ordered by v so that if $v_1<v_2$, then

$$P_{v_1}\{Y_n(r_1)\le y\}\ge P_{v_2}\{Y_n(r_1)\le y\}.$$

While the exact value of $M_n(v)$ and hence $P_v\{Y_n(r_1)=y\}$ are difficult to find even when n is small and D_n is a regular region, Saunders and Funk [9] have given approximations for the distribution of $Y_n(r_1)$ which can be used when $v\le 0$, n is large and the points are sparse in D_n. To define sparseness let $S_r(z)$ denote a γ-dimensional sphere of radius r having center at z, and note for any $r>0$

$$E_0\{Y_n(r)\}=\binom{n}{2}\left[\frac{\mu(S_r(z_n))\mu(I_n)}{[\mu(D_n)]^2}+\int_{B_n}\frac{\mu(S_r(x)\cap D_n)}{[\mu(D_n)]^2}\,d\mu\right].$$

Here E_0 denotes expectation in the i.i.u.d. case of $v=0$,

$$I_n=\{x: x\in D_n \text{ and } S_r(x)\subseteq D_n\},$$

z_n is some point in I_n, and B_n (the r-boundary of D_n) is the set of points in D_n but not in I_n. Sparseness is said to hold if both (1.1) and (1.2) below hold for all $r>0$:

$$\lim_{n\to\infty}\binom{n}{2}\frac{\mu(S_r(z_n))\mu(I_n)}{[\mu(D_n)]^2}=\lambda r^{\gamma}\quad(\lambda>0) \tag{1.1}$$

and

$$\lim_{n\to\infty}\binom{n}{2}\int_{B_n}\frac{\mu(S_r(x)\cap D_n)}{[\mu(D_n)]^2}\,d\mu=0. \tag{1.2}$$

Less formally (1.1) requires that $\mu(D_n)/n^2$ has roughly a constant value. This requirement is essentially equivalent to requiring that the region is sufficiently large so that even though D_n contains many points it is a rare event to find two points close to each other. That is, the points are sparse in the region. The second condition, (1.2), is primarily needed to make the boundary effects negligible in proof so that the limiting mean function of $Y_n(r)$ is tractable for approximate inference techniques concerning v and r_0. Note that by making a suitable transformation it is possible to interpret these conditions and the results below for the situation where D_n is a fixed region and $Y_n^*(r)$ is the short interpoint distance function $Y_n^*(r) = Y_n(r/n)$.

Potential applications where sparseness should hold include the situation where the points represent the locations of rare plants in a large study region, or the nesting or lair location of solitary and rare biological species. Rules of thumb for deciding on the goodness of the Poisson approximations when sparseness does not precisely hold have been discussed in [9] and appear in (3.2) below.

The main result in [9] follows.

Theorem 1.1. *If sparseness holds and $v < 0$, then as $n \to \infty$ the interpoint distance function $\{Y_n(r)\colon 0 \leq r \leq r_{00}\}$ converges weakly to a non-homogeneous Poisson process $\{Y(r)\colon 0 \leq r \leq r_{00}\}$ having mean function*

$$E_v\{Y(r)\} = \begin{cases} \lambda r^\gamma \mathrm{e}^v, & 0 \leq r \leq r_1, \\ \lambda r_1^\gamma \mathrm{e}^v + (r^\gamma - r_1^\gamma), & r_1 < r \leq r_{00}, \end{cases}$$

where $r_{00} > r_1$ is some radius beyond which influence is inconceivable.

When $v > 0$ it is impossible to establish a result like Theorem 1.1 even under the sparseness conditions, and Saunders and Funk [9] have established that for $v > 0$

$$\lim_{n \to \infty} P_v\{Y_n(r) \leq y\} = 0$$

for all $r > 0$ and all $y = 0, 1, 2, \ldots$. In this note we introduce a modification of the Strauss model which makes it possible to establish a result like Theorem 1.1 even when $v > 0$. This modification consists of imposing the simple restriction that there be a γ-dimensional sphere of radius $r_0 < r_1$ about each point where no other point can be located. Thus r_0 could represent a minimal interpoint distance required for survival or room between objects such as large trees or animal lairs. The physical size of such objects precludes a crowding of many points very close to each other. The introduction of r_0 into the Strauss model is exactly what is needed to determine the asymptotic behaviour of $Y_n(r)$ when $v > 0$.

When $v = 0$ this modified model is exactly the hard-core model considered by Matérn [5] and when $v > 0$ this model has been suggested by a spectral analysis of redwood seedling data performed by Ripley [4]. In all cases the modified model can be obtained by conditioning on the events $\{Y_n(r_0) = 0\}$ in the Strauss model, and we use this fact to derive results analogous to Theorem 1.1 in the next section.

These limiting results have applications to statistical problems for spatial data and in particular we use these results to study the power functions of some tests of spatial randomness in Section 3.

2. Limiting results for the hard-core modification of the Strauss model

The hard-core modification of the Strauss model assumes that the joint density has the form

$$f^*(x_1,\dots,x_n)=\begin{cases} e^{vy_n(r_1)}/M_n^*(v) & \text{if } x_1\in D_n,\dots,x_n\in D_n,\ y_n(r_0)=0,\\ 0 & \text{otherwise}\end{cases} \tag{2.1}$$

where $M_n^*(v)$ is the conditional moment generating function of $Y_n(r_1)$ given the event $H_n=\{Y_n(r_0)=0\}$ when $X_{1n},\dots,X_{nn}$ are i.i.u.d. Thus when $v=0$ the hard-core modification assumes that the locations are essentially random but each point has a hard-core sphere of radius r_0 into which no other point may be placed.

Using $W_n(r)$ to denote the random variable $Y_n(r)$ conditioned on H_n, Theorem 2 of Saunders and Funk [9] shows that when sparseness holds,

$$\begin{aligned}\lim_{n\to\infty} P_0\{W_n(r)=k\} &= \frac{\lim_{n\to\infty} P_0\{Y_n(r)=k,\ Y_n(r_0)=0\}}{\lim_{n\to\infty} P_0\{Y_n(r_0)=0\}}\\ &=[\lambda(r^\gamma-r_0^\gamma)]^k\exp(-\lambda(r^\gamma-r_0^\gamma))/k! \end{aligned}\tag{2.2}$$

for $k=0,1,2,\dots$. Thus $W_n(r)$ has a Poisson limiting distribution when $v=0$ and $r\geq r_0$.

Note that when $v\neq 0$ the moment generating function of $W_n(r)$ is of the form

$$E_v\{\exp(tW_n(r))\}=\frac{E_0\{\exp(tW_n(r)+vW_n(r_1))\}}{E_0\{\exp(vW_n(r_1))\}}. \tag{2.3}$$

Using Theorem 2 of Saunders and Funk [9] and an argument similar to the one above it is also possible to show that if $v=0$ and sparseness holds, then for any choice $r_0\leq r(1)<\cdots<r(m)$ the random vector

$$(W_n(r(1)),\ W_n(r(2))-W_n(r(1)),\dots,\ W_n(r(m))-W_n(r(m-1)))$$

converges in distribution to a vector $(W(r(1)),\dots,W(r(m))-W(r(m-1)))$ of independent Poisson random variables having mean values

$$E_0\{W(r(i))-W(r(i-1))\}=\lambda(r(i)^\gamma-r(i-1)^\gamma)$$

for $i=1,\dots,m$ with $r(0)\equiv r_0$. This and (3) can be used to show that if $v<0$, then the moment generating function of $W_n(r)$ converges for all $t<-v$ to the moment generating function of a Poisson random variable showing that if $v<0$, then $W_n(r)$ has a Poisson limiting distribution. The mean of this limiting distribution can be shown to be

$$\lim_{n\to\infty} E_v\{W_n(r)\} = \begin{cases} e^v\lambda(r^\gamma - r_0^\gamma) & \text{if } r_0 \le r < r_1, \\ e^v\lambda(r_1^\gamma - r_0^\gamma) + (r^\gamma - r_1^\gamma) & \text{if } r > r_1. \end{cases} \tag{2.4}$$

When $v > 0$ the convergence of the distributions in (2.2) is not sufficient (see, e.g., [2]) to imply the convergence of the generating functions on the right-hand side of (2.3) to the generating functions of the limiting distributions. Note, however, that if these generating functions did converge to the generating functions of the limiting Poisson distributions, then this would show that $W_n(r)$ has a limiting distribution with mean given by (2.4) even when $v > 0$. To show that these generating functions converge to the generating functions of the limiting Poisson distributions it is necessary and sufficient (see [10]) to establish uniform bounds for these generating functions. Specifically, we need to show that for each fixed $t > 0$ and $r \geq r_0$ there is a constant free of n such that

$$E_0\{e^{tW_n(r)}\} \leq C(t, r) < \infty.$$

To indicate the existence of these constants we prove the following result.

Lemma 2.1. *Let D_n be a square subregion of $\mathbb{R}^2$ having diagonals of length nr_0. For fixed $r \geq r_0$ and $t > 0$ there is a constant $C(t, r)$ not depending on n such that if sparseness holds* ($\lambda = 1$), *then*

$$E_0\{e^{tW_n(r)}\} \leq C(t, r) < \infty.$$

Proof. Partition D_n into n^2 square subregions having diagonals of length r_0 and denote these regions by $D_n(\boldsymbol{\omega})$ where $\boldsymbol{\omega}$ is the center of the region. Suppose n i.i.u.d. points $X_{1n}, \ldots, X_{nn}$ are chosen in D_n, and let $N_{\boldsymbol{\omega}}$ denote the number of these n points which fall in $D_n(\boldsymbol{\omega})$. Further define the event

$$A_n = \{\max_{\boldsymbol{\omega}} D_n(\boldsymbol{\omega}) = 1\}$$

and observe that $H_n \subset A_n$ which implies that for $r_0 < r$

$$P_0\{H_n \cap \{Y_n(r) = k\}\} \leq P_0\{A_n \cap \{Y_n(r) = k\}\}$$

or equivalently

$$P_0\{W_n(r) = k\} \leq c_n P_0\{Y_n(r) = k \mid A_n\}.$$

Here as $n \to \infty$

$$\lim c_n = \frac{\lim P_0\{A_n\}}{\lim P_0\{H_n\}} \leq \frac{1}{\lim P\{H_n\}} = e^{\pi r_0^2} = C(r_0).$$

Now consider the random variable

$$V_n(r) = \sum_{\|\omega - \omega'\| \leq r + r_0} N_{\omega} N_{\omega'}$$

and observe that, given the event A_n, $Y_n(r) \leq V_n(r)$. Combining the results above it follows easily that

$$E_0\{e^{tW_n(r)}\} \leq C(r_0)E_0\{e^{tY_n(r)} | A_n\} \leq C(r_0)E_0\{e^{tV_n(r)} | A_n\}.$$

Thus the proof would be complete provided $E_0\{e^{tV_n(r)} | A_n\}$ converged when $n \to \infty$. To see this is the case note that if $v = 0$, then given A_n, the variable $V_n(r)$ has exactly the same distribution as if each of the n points were selected at random without replacement from the lattice consisting of the centers of the subregions $D_n(\omega)$. In [10] it has been shown that under these conditions the generating functions of statistics like $V_n(r)$ do in fact converge to the generating functions of a limiting distribution which is a Poisson distribution. Thus a straightforward application of Lemma 4 in [10] now implies the existence of the constants.

While the results in [10] are given only for the two-dimensional rectangular lattice these results extend easily to other dimensional lattices provided the lattice boundary is small compared to the number of points in the lattice. These extended results can be used in conjunction with arguments almost identical to those of the lemma above to establish the convergence of the generating functions in the right-hand side of (2.3) provided sparseness holds. We summarize these discussions as the following result.

Theorem 2.2. *If sparseness holds, then for any real v the random variables $W_n(r)$ converge in distribution to a Poisson random variable $W(r)$ with mean given by* (2.4).

This result can be easily extended to show that the finite dimensional distributions of the collection of random variables $\{W_n(r): r_0 \leq r \leq r_{00}\}$ converge in distribution to the finite dimensional distributions of a Poisson process having the mean function given by (2.4). Thus we can consider weak convergence of the processes $\{W_n(r): r_0 \leq r \leq r_{00}\}$ on D[0, 1] with the Skorohod topology (with the obvious radius scale change). By using the fact that $W_n(r) \leq W_n(r')$ for $r \leq r'$ and some rather tedious arguments which justify the application of Theorem 15.1 and Theorem 15.3 of [1] the following result can be shown.

Theorem 2.3. *If sparseness holds, then, as $n \to \infty$, the processes $\{W_n(r): r_0 \leq r \leq r_{00}\}$ converge weakly to a Poisson process $\{W(r): r_0 \leq r \leq r_{00}\}$ having mean function given by* (2.4).

It is possible to establish results similar to Theorem 2.3 when the hard-core components have different shapes and sizes or if the number of points in D_n is a random variable, provided these deviations from the model discussed here are compatible with sparseness and effectively bound the number of points which can occur in any region of specified size.

3. Applications to tests for randomness

The limiting result in Theorem 2.3 can be used to construct tests of hypotheses for the model parameters and to compute the approximate power of tests for spatial randomness. Some examples follow.

To begin consider the situation where n points $X_1, \ldots, X_n$ are distributed on the two-dimensional unit square $[0,1]\times[0,1]$. Letting d_{in} denote the ith smallest interpoint distance between these points, consider rejecting the null hypothesis of randomness when large values of d_{in} are observed. In applying our results to this situation we consider radii of the form $r_{0n} = r_0/n$ and $r_{1n} = r_1/n$, and test the null hypothesis of randomness defined by H_0: $r_0 = 0$ and $v = 0$ against the modified Strauss model. The power function of the critical region $d_{in} > r/n$ is

$$\pi(r_0, r_1, v, d_{in}) = P_v\{d_{in} > r/n \mid r_{0n} = r_0/n,\ r_{1n} = r_1/n\}.$$

To approximate this function we take $X_{1n} = nX_1, \ldots, X_{nn} = nX_n$ so that $D_n = [0, n]\times[0, n]$ and assume n is large enough while r_1 is small enough to ensure that the sparseness conditions hold. Then using Theorem 2.3 we obtain

$$\pi(r_0, r_1, v, d_{in}) \approx \exp(-\beta_r) \sum_{j=0}^{i-1} \beta_r^j/j! \tag{3.1}$$

where $\beta_r = E_v\{W(r)\}$ which is defined by (2.4). For the randomness model Ripley and Silverman [8] used the approximation (3.1) to obtain approximate critical values for the statistics $d_{1,n}$, $d_{3,n}$ and $d_{5,n}$, but to study the power of these test statistics against the hard core model of Matérn or the repulsion version of the Strauss model they resorted to a simulation study.

In Table 1 we compare the power approximations obtained by using simulations in [8] with the power approximations obtained by using (3.1) with $n = 50$ and $\lambda \approx \binom{n}{2}\pi$. In addition we present the ratio κ_3/κ_1 for each table simulation and note that provided $\kappa_3/\kappa_1 < 2.0$ the numerical simulation and approximation results are quite close. The quantity κ_3/κ_1 represents the ratio of the first to the third cumulant of $Y_n(r)$ under the randomness model. In [9] we have shown that a rule of thumb for sparseness to hold is that $\kappa_3/\kappa_1 < 2.0$, and that if we ignore boundary effects, then, with D_n representing the unit square,

$$\kappa_3/\kappa_1 \approx (1-A)(1-2A) + 2(n-2)(1-3\sqrt{3}/4\pi - A)A, \tag{3.2}$$

where $A = \pi r^2$.

The power of tests based on other functions of the smaller interpoint distances can be approximated in a similar manner not only for the hard-core and the Strauss model with $v < 0$ but for any of the models covered by Theorem 2.3.

Another method of testing for randomness is to use statistics which examine the behaviour of $Y_n(r)$ as a function of r. One statistic of this form, suggested in [3], is

$$T_1 = \sup_{0 \le r \le r_2} \left| \frac{Y_n(r)}{E_0\{Y_n(r_2)\}} - \frac{E_0\{Y_n(r)\}}{E_0\{Y_n(r_2)\}} \right|.$$

Table 1. A comparison of the approximation to the power of 5% level tests for randomness obtained by the asymptotic formula (3.1) with the approximation obtained in [8] by simulation (in parentheses).

		Hard-core model $r_0 = R$, $v = 0$			Strauss model $r_0 = 0$, $r_1 = R$					
					$v = -2.3$			$v = -1.2$		
R	κ_3/κ_1	d_1	d_3	d_5	d_1	d_3	d_5	d_1	d_3	d_5
0.02	1.07	0.23 (0.27)	0.15 (0.15)	0.12 (0.17)	0.20 (0.17)	0.13 (0.11)	0.11 (0.09)	0.15 (0.16)	0.11 (0.08)	0.10 (0.17)
0.03	1.15	1.0 (1.0)	0.46 (0.47)	0.33 (0.38)	0.74 (0.73)	0.38 (0.39)	0.28 (0.33)	0.41 (0.39)	0.26 (0.30)	0.20 (0.23)
0.04	1.27	1.0 (1.0)	1.0 (1.0)	0.82 (0.81)	0.74 (0.73)	0.96 (0.86)	0.70 (0.75)	0.41 (0.33)	0.68 (0.65)	0.47 (0.46)
0.06	1.59	1.0 (1.0)	1.0 (1.0)	1.0 (1.0)	0.74 (0.63)	0.97 (0.95)	1.0 (1.0)	0.41 (0.32)	0.71 (0.65)	0.86 (0.80)
0.08	2.03	1.0 (1.0)	1.0 (1.0)	1.0 (1.0)	0.74 (0.55)	0.97 (0.89)	1.0 (0.99)	0.41 (0.25)	0.71 (0.68)	0.86 (0.72)

Consider modifying T_1 as follows. Suppose we are interested in testing randomness versus the clustering version of the modified Strauss model; that is, suppose we are interested in testing H_0: $v = 0$ and $r_0 = 0$ versus H_1: $v > 0$ and $r_0 > 0$ with v, r_1 and r_0 unspecified. If $L_{1n}^* < r_2$ where $L_{1n}^* = \inf\{r\colon Y_n(r) = 1\}$, then an approximate test of H_0 can be constructed by first conditioning on the events $\{L_{1n}^* = l\}$ and $\{Y_n(r_2) = y_n(r_2)\}$, and then computing the statistic

$$T_2 = \sup_{0 \le t \le 1} \{U_n(t) - 1\}$$

where $U_n(t) = \{Y_n(r(t)) - 1\}/\{y_n(r_2) - 1\}$ with $r(t) = (t(r_j^2 - l^2) + l^2)^{1/2}$. If H_0 is true and $y_n(r_2) > 1$, then the result of Theorem 2.3 can be used to show that T_2 has approximately the same conditional distribution as the Kolmogorov–Smirnov statistic. Furthermore, for any fixed values of $r_1 > r_0 > 0$ and $v > 0$ the approximate power of this conditional test can be computed by using Theorem 2.3 and the iterative methods of Noé and Vandewiele [6]. Finally, if we wish to test the hypothesis of randomness versus only the hard core model of Matérn, the test statistic T_2 can be simplified by conditioning solely on $Y_n(r_2)$; that is, by setting $U_n(t) = Y_n(tr_2)/y_n(r_2)$.

4. Conclusions

Note that the limiting result of Theorem 2.3 can be extended to give Poisson approximations for other measures of clustering and similar approximations made for the critical values and power of tests based on these measures. One measure for which this extension can be made is the process $\{Z_n(r)\colon 0 \le r \le r_2\}$ where $Z_n(r)$ is the number of nearest neighbor distances less than r, and provided sparseness holds, the behavior of tests based on this process can equally well be approximated using Poisson processes results.

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